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Self and mutual interaction of electromagnetic waves in a magnetoplasma considering the dependence of electron density on electron temperature†

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Abstract. The author has obtained an expression for the change in electron density in a magnetoplasma arising due to the change in the electron temperature when two electromagnetic waves of frequency ω_1 and ω_2 propagate through it. This expression for the modified electron density is used to derive expressions for the current density due to the electric vectors of the two waves. The expressions thus obtained are substituted in the general wave equation; the solutions of the wave equation are used to investigate the non-linear self and mutual interaction of the two waves. Some numerical results have been presented in the form of tables to illustrate the dependence of the amplitudes of the waves, for both the processes of mutual and self interaction of the waves, on relevant parameters; these calculations have been carried out for a very particular case when the electron density is assumed to be governed by Saha's equation corresponding to the electron temperature. These numerical calculations suggest that it is important to take account of the variation in the electron density (arising due to changes of ionization and de-ionization rates) caused by the waves propagating through a plasma. Using the theory developed in this paper, one can obtain the dependence of electron density on temperature from a study of the self and mutual interaction.

1. Introduction

It is well known that an electromagnetic wave propagating through a plasma changes the properties of the medium, which results in the well-known phenomena of self interaction and mutual interaction (e.g. Luxembourg effect) with other waves simultaneously passing through the plasma. These non-linear interaction phenomena occur because of the change in the density and the complex mobility of the electrons. The change in the complex electron mobility (which depends on the electron collision frequency) has been incorporated in the analysis by many earlier workers, e.g. Ginzburg and Gurevich (1960) and Sodha and Palumbo (1963). In their excellent review of the subject, Ginzburg and Gurevich (1960) have also pointed out the desirability of taking into account the change in the electron density due to the passage of the high-intensity electromagnetic waves; this change in the electron density occurs because of the change in the electron temperature caused by the wave.

In this communication the author has analysed the non-linear interaction of electromagnetic waves in a plasma initially in thermal equilibrium, so that the electron temperature T_e is equal to the gas temperature T_0. Because of the ohmic heating caused by the propagation of an electromagnetic wave, the electron temperature changes, and hence the electron density also changes; this happens because the ionization and de-ionization rates depend upon the electron temperature. Based on these and other (change in complex mobility) considerations, the author has also derived an expression for the current density in a magnetoplasma when two electromagnetic waves are propagating along the z direction. It is seen that the component of the current density, alternating with the frequency of either wave, depends upon the amplitude of both the electromagnetic waves and involves a term $\partial N_e/\partial T_e$ which expresses the dependence of electron density on electron temperature. The

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above expression for the current density in a magnetoplasma has been substituted in the general wave equation and the resulting set of non-linear differential equations has been solved by the method of successive approximations. The differential equation for the propagation of each of the waves contains terms depending upon the amplitudes of the first wave and the second wave, thus illustrating the phenomena of self interaction and mutual interaction of the two waves. These solutions of the wave equation have been used to investigate both these phenomena. The analysis has been applied to a very particular case when (following Kerrebrock 1964, and others in the field of non-equilibrium ionization) the electron density is governed by Saha's (1921) equation corresponding to the electron temperature. Some numerical calculations have been carried out, on an IBM 1620 computer, to investigate the dependence of the amplitude of the first wave on the relevant parameters of the medium for the processes of self and mutual interaction of the two waves. It is seen that, for a typical case, the contribution of taking into account the change in electron density amounts to about 20% of the non-linear part of the complex amplitude of the waves.

2. Electron density in a magnetoplasma

If one considers a uniform plasma subjected to an electric field of such a strength that the electron temperature T_e is only slightly greater than the gas temperature T_0†, then the electron density N_e (at the temperature T_e) may be expressed in terms of the electron density N_0 (at the temperature T_0) by a Taylor expansion of the form

$$N_e = N_0 + \left(\frac{\partial N_e}{\partial T_e}\right)_{T_e=T_0} (T_e - T_0). \tag{1}$$

This change in electron density is caused by the change in ionization and de-ionization rates due to the change in the electron temperature when a wave propagates through the plasma.

If a plasma is subjected to a magnetic field B along the z axis and two electromagnetic waves of frequencies ω_1 and ω_2, i.e.

$$\begin{aligned} E_x &= E_{1x}\exp(i\omega_1 t) + E_{2x}\exp(i\omega_2 t) \\ E_y &= E_{1y}\exp(i\omega_1 t) + E_{2y}\exp(i\omega_2 t) \end{aligned} \tag{2}$$

then this corresponds to the physical situation of two electromagnetic waves travelling along the direction of the external magnetic field. The energy balance equation of the magnetoplasma may be written as (Sodha 1965)

$$\frac{\boldsymbol{J}\,.\,\boldsymbol{E}}{N_e} = \frac{2mG}{M}\langle(\tfrac{1}{2}mv^2 - lkT_0)\nu\rangle \tag{3}$$

where $\boldsymbol{J}$ is the total current density, $\nu(v)$ is the electron collision frequency, m and M are the masses of the electron and the heavy particles respectively, v is the random velocity of the electron, k is Boltzmann's constant, G is a factor which gives the fractional energy change (taking inelastic collisions into account) and l is a constant to be determined. The left-hand side of the above equation represents the average power received by an electron from the field while the right-hand side gives the average energy lost by the electron per unit time by collisions. Knowing that the average energy lost by an electron owing to collisions is zero when the electron temperature T_e is equal to the gas temperature T_0, the constant l can be determined. Assuming that the electron collision frequency can be represented by

$$\nu = av^s \tag{4}$$

and that the electrons obey a Maxwellian distribution of velocities, the right-hand side of equation (3) can be evaluated.

† This is true when $\{Me^2E_{00}{}^2/6m^2(\nu^2+\omega^2)kT_0\} \ll 1$ where the symbols have their usual meanings.

The complex mobility $\mu_{\parallel}$ of an electron subjected to an electric field of frequency ω along the direction of the electric field in a plasma, and the mobility $\mu_{\perp}$ in a perpendicular direction, are given by (Sodha and Palumbo 1963)

$$\mu_{\parallel} = \frac{e}{3m}\left\langle \frac{1}{v^2}\frac{\mathrm{d}}{\mathrm{d}v}\left\{\frac{(\nu+\mathrm{i}\omega)v^3}{(\nu+\mathrm{i}\omega)^2+\omega_B{}^2}\right\}\right\rangle$$

and

$$\mu_{\perp} = \frac{e}{3m}\left\langle \frac{1}{v^2}\frac{\mathrm{d}}{\mathrm{d}v}\left\{\frac{\omega_B v^3}{(\nu+\mathrm{i}\omega)^2+\omega_B{}^2}\right\}\right\rangle \tag{5}$$

where e is the electronic charge and $\omega_B = (eB/mc)$ is the electron gyrofrequency. It may be remarked that equations (5) are non-linear in character; the non-linearity enters through averaging over a distribution which depends upon the electric vector. In the present analysis the averaging process is carried out by assuming that the electrons satisfy a Maxwellian distribution of velocities corresponding to a temperature higher than that of the gas. In most physical situations of interest in a plasma $\omega \gg \nu$, hence, knowing that

$$J_x = e(N_e\mu_{\parallel}E_x - N_e\mu_{\perp}E_y)$$

and

$$J_y = e(N_e\mu_{\perp}E_x + N_e\mu_{\parallel}E_y) \tag{6}$$

a solution of (3) would yield an expression for $(T_e-T_0)/T_0$. Using this solution for T_e-T_0 in equation (1), we obtain the following expression for the electron density N_e for the case when the frequency ω_1 of the first wave equals the gyrofrequency of electrons:

$$\begin{aligned} N_e = N_0[1+b\{&P_1(E_{1x}\tilde{E}_{1x}+E_{1y}\tilde{E}_{1y})+L_1(E_{1x}\tilde{E}_{1y}-E_{1y}\tilde{E}_{1x}) \\ &+P_2(E_{2x}\tilde{E}_{2x}+E_{2y}\tilde{E}_{2y})+L_2(E_{2x}\tilde{E}_{2y}-E_{2y}\tilde{E}_{2x})\}] \end{aligned} \tag{7}$$

where

$$\left.\begin{aligned} b &= \left\{Me^2\left(\frac{\partial N_e}{\partial T_e}\right)_{T_e=T_0}\frac{1}{6m^2k\nu_0GT_0N_0\Gamma(5/2+s/2)}\right\}, \qquad \nu_0 = a\left(\frac{2kT_0}{m}\right)^{s/2} \\ P_1 &= \frac{\Gamma(5/2-s/2)}{2\nu_0}-\frac{3\sqrt{\pi}\mathrm{i}}{16\omega_1}, \qquad L_1 = \frac{3\sqrt{\pi}}{16\omega_1}-\frac{\mathrm{i}\Gamma(5/2-s/2)}{2\nu_0} \\ P_t &= \frac{1}{(\omega_B{}^2-\omega_t{}^2)^2}\left[\nu_0{}^3\Gamma\left(\frac{5}{2}+\frac{3s}{2}\right)+(\omega_t{}^2+\omega_B{}^2)\nu_0\Gamma\left(\frac{5}{2}+\frac{s}{2}\right)\right. \\ &\quad \left.+\mathrm{i}\omega_t\left\{\frac{3\sqrt{\pi}}{4}(\omega_B{}^2-\omega_t{}^2)-\nu_0{}^2\Gamma\left(\frac{5}{2}+s\right)\right\}\right] \\ \text{and} \quad L_t &= \frac{\omega_B}{(\omega_B{}^2-\omega_t{}^2)^2}\left\{\frac{3\sqrt{\pi}}{4}(\omega_B{}^2-\omega_t{}^2)+\nu_0{}^2\Gamma\left(\frac{5}{2}+s\right)-2\mathrm{i}\omega_t\nu\Gamma_0\left(\frac{5}{2}+\frac{s}{2}\right)\right\} \end{aligned}\right\} \tag{8}$$

where t takes the value 1 and 2 for the two waves. For a general case (when $\omega_1 \neq \omega_B \neq \omega_2$) the form of equation (7) remains the same but the coefficients are given by the last two equations of (8). The terms containing the time-dependent components of the electron temperature have not been considered in equation (7) because the electron density can not follow the instantaneous changes in the electron temperature at high frequencies.

3. Current density in a magnetoplasma

The electric vector in a magnetoplasma (when the magnetic field acts along the z direction) in the presence of two electromagentic waves propagating along the z direction, is given by equation (2). Using equations (5), (6), (7) and (2), one obtains the following expression for the component of the current density due to the wave of frequency ω_1 at its gyro-resonance ($\omega_1 = \omega_B$)

$$
\begin{aligned}
J_{1x}+\mathrm{i}J_{1y} = \Big[&A_1{}^+(E_{1x}+\mathrm{i}E_{1y})+\alpha\Big[C_1\{P_1(\epsilon_{1x}\tilde{\epsilon}_{1x}+\epsilon_{1y}\tilde{\epsilon}_{1y})+L_1(\epsilon_{1x}\tilde{\epsilon}_{1y}-\epsilon_{1y}\tilde{\epsilon}_{1x})\\
&+P_2(\epsilon_{2x}\tilde{\epsilon}_{2x}+\epsilon_{2y}\tilde{\epsilon}_{2y})+L_2(\epsilon_{2x}\tilde{\epsilon}_{2y}-\epsilon_{2y}\tilde{\epsilon}_{2x})\}(E_{1x}+\mathrm{i}E_{1y})\\
&+\frac{B_1P_1}{2}(\epsilon_{1x}\epsilon_{1x}+\epsilon_{1y}\epsilon_{1y})(\tilde{E}_{1x}+\mathrm{i}\tilde{E}_{1y})+\frac{B_2}{2}[\{(P_1+P_2)(\epsilon_{1x}\epsilon_{2x}+\epsilon_{1y}\epsilon_{2y})\\
&+(L_1-L_2)(\epsilon_{1x}\epsilon_{2y}-\epsilon_{1y}\epsilon_{2x})\}(\tilde{E}_{2x}+\mathrm{i}\tilde{E}_{2y})+\{(P_1+P_2)(\epsilon_{1x}\tilde{\epsilon}_{2x}+\epsilon_{1y}\tilde{\epsilon}_{2y})\\
&+(L_1-\tilde{L}_2)(\epsilon_{1x}\tilde{\epsilon}_{2y}-\epsilon_{1y}\tilde{\epsilon}_{2x})\}(E_{2x}+\mathrm{i}E_{2y})]\Big]_2\Big]_1\exp(\mathrm{i}\omega_1 t)
\end{aligned} \tag{9}
$$

where $\alpha = (Me^2E_{00}{}^2/6m^2kT_0G\omega_0{}^2)$, $\epsilon = E/E_{00}$, E_{00} is an arbitrary field, ω_0 is an arbitrary frequency, $\omega_p{}^2 = (4\pi N_0e^2/m)$ and

$$
\left.
\begin{aligned}
A_1{}^+ &= \frac{\omega_p{}^2\Gamma(5/2-s/2)}{3\pi^{3/2}\nu_0}, \qquad c_1 = F_1+B_1\\
F_1 &= \frac{T_0}{N_0}\left(\frac{\partial N_e}{\partial T_e}\right)_{T_e=T_0}\frac{\omega_0{}^2A_1{}^+}{\nu_0\Gamma(5/2+s/2)}, \qquad B_1 = -\frac{\omega_p{}^2\omega_0{}^2s\Gamma(5/2-s/2)}{6\pi^{3/2}\nu_0{}^2\Gamma(5/2+s/2)}\\
B_t &= \frac{\omega_p{}^2\omega_0{}^2}{3\pi^{3/2}\nu_0\Gamma(5/2+s/2)(\omega_B{}^2-\omega_t{}^2)^2}\left[\frac{3s}{2}\nu_0{}^3\Gamma\left(\frac{5}{2}+\frac{3s}{2}\right)+\frac{s}{2}(\omega_t{}^2+\omega_B{}^2)\nu_0\Gamma\left(\frac{5}{2}+\frac{s}{2}\right)\right.\\
&\left.-\mathrm{i}\omega_t\nu_0{}^2s\Gamma\left(\frac{5}{2}+s\right)+\mathrm{i}\omega_Bs\left\{\nu_0{}^2\Gamma\left(\frac{2}{5}+s\right)-\mathrm{i}\omega_t\nu_0\Gamma\left(\frac{5}{2}+\frac{s}{2}\right)\right\}\right]
\end{aligned}
\right\} \tag{10}
$$

where $t = 1, 2$ for the two frequencies. For a general case (when $\omega_1 \neq \omega_B \neq \omega_2$) the form of equation (9) remains the same; the coefficients have the same values as given by equations (10) except that B_1 is now obtained from the last equation of (10) and $A_1{}^+$ takes the value

$$
\begin{aligned}
A_1{}^+ = \frac{\omega_p{}^2}{3\pi^{3/2}(\omega_B{}^2-\omega_1{}^2)^2}\Big[&\nu_0{}^3\Gamma\left(\frac{5}{2}+\frac{3s}{2}\right)+(\omega_1{}^2+\omega_B{}^2)\nu_0\Gamma\left(\frac{5}{2}+\frac{s}{2}\right)\\
&+2\omega_B\omega_1\nu_0\Gamma\left(\frac{5}{2}+\frac{s}{2}\right)+\mathrm{i}(\omega_1+\omega_B)\left\{\frac{3\sqrt{\pi}}{4}(\omega_B{}^2-\omega_1{}^2)-\nu_0{}^2\Gamma\left(\frac{5}{2}+s\right)\right\}\Big].
\end{aligned} \tag{11}
$$

A similar expression for the component $J_{2x}+\mathrm{i}J_{2y}$ of the current density due to the second wave can be obtained by replacing ω_1 by ω_2, and E_{1x}, E_{1y}, by E_{2x}, E_{2y} respectively in the above equations. The expressions for $J_{1x}-\mathrm{i}J_{1y}$ and $J_{2x}-\mathrm{i}J_{2y}$ can be written down by replacing $+\mathrm{i}$ by $-\mathrm{i}$ in the corresponding expressions obtained above.

4. Propagation of the electromagnetic waves in a magnetoplasma

Consider the x and y components of the general wave equation of electromagnetic waves in a neutral medium: multiplying the latter by $+\mathrm{i}$ and adding it to the former, one obtains the following equation for the propagation of the extraordinary mode of the wave travelling along the z direction

$$\frac{\partial^2(E_x+\mathrm{i}E_y)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2(E_x+\mathrm{i}E_y)}{\partial t^2}+\frac{4\pi}{c^2}\frac{\partial}{\partial t}(J_x+\mathrm{i}J_y) \tag{12}$$

which may be written in a dimensionless form

$$\frac{\partial^2(\epsilon_x+\mathrm{i}\epsilon_y)}{\partial \xi^2} = \frac{1}{\omega_0^2}\frac{\partial^2(\epsilon_x+\mathrm{i}\epsilon_y)}{\partial t^2}+\frac{4\pi}{\omega_0^2}\frac{\partial}{\partial t}\left(\frac{J_x+\mathrm{i}J_y}{E_{00}}\right) \tag{13}$$

by putting $\xi = (\omega_0 z/c)$ where c is the velocity of light in vacuum.

The expression for the total current density in a magnetoplasma subjected to the electric vector of equation (2) can be written with the help of the results obtained in the last section. Substituting this expression in equation (13) and equating the time-dependent terms of the same frequency on both the sides, one obtains

$$\frac{\partial^2(\epsilon_{1x}+\mathrm{i}\epsilon_{1y})}{\partial \xi^2} = -(\beta_1{}^+)^2(\epsilon_{1x}+\mathrm{i}\epsilon_{1y})+\frac{\alpha K_1{}^+}{E_{00}}[D] \tag{14}$$

where $[D]$ stands for the expression contained in the second bracket $[_2\ ]_2$ of (9),

$$K_1{}^+ = \frac{4\pi\mathrm{i}\omega_1}{\omega_0}$$

and

$$(\beta_1{}^+)^2 = \frac{\omega_1{}^2}{\omega_0{}^2}\left(1-\frac{4\pi\mathrm{i}A_1{}^+}{\omega_1}\right) = (-n_1+\mathrm{i}k_1)^2. \tag{15}$$

An equation for the propagation of the extraordinary mode of the second wave can be similarly obtained by substituting the proper expression for the current density component from the last section. The corresponding equations for the propagation of the ordinary components $\epsilon_x-\mathrm{i}\epsilon_y$ can be written down in a similar manner where $\beta_1{}^+$ is replaced by $\beta_1{}^-$. Because the wave equation for the first wave contains both the electric vector and current density, which in turn depends on its own amplitude and the amplitude of the second wave, its propagation is influenced by the wave itself and the second wave, and hence the processes of self and mutual interaction take place.

These non-linear differential equations, like (14), can be solved by the method of successive approximation of the type used by earlier workers, namely, Epstein (1962), Sodha and Palumbo (1962). The constants of integration can be evaluated from the boundary conditions

$$\left.\begin{array}{ll}\epsilon_{1x} = \epsilon_{1x0}, & \epsilon_{1y} = \epsilon_{1y0}\\ \epsilon_{2x} = \epsilon_{2x0}, & \epsilon_{2y} = \epsilon_{2y0}\end{array}\right\}\text{at } \xi = 0 \tag{16}$$

and the radiation condition, i.e. that all the components of the electric vector vanish at $\xi \to \infty$.

5. Discussion

We have taken the electron collision frequency ν in equations (5) to correspond to elastic collisions only. This is a good approximation because, for cases of practical interest, the elastic collisions are much more frequent than the inelastic collisions.

Equations (10) illustrate that, all the coefficients (arising because of both fields) being directly proportional to the square of the plasma frequency ω_p, the magnitudes of all the components of the current density increase directly with the electron density. The solution of the non-linear differential wave equations (14) gives an expression for the amplitude of either of the waves, taking into account the dependence of N_e on T_e. It is suggested, therefore, that a study of self and mutual interaction of the waves propagating through a plasma may provide a means to obtain the dependence of electron density on electron temperature.

Let us apply the above analysis to a particular case when the electron density in a slightly ionized gas is given by Saha's equation corresponding to the electron temperature (Kerrebrock 1964):

$$N_e = C_1 T_e^{3/4} \exp\left(-\frac{U}{2kT_e}\right) \tag{17}$$

where U is the ionization potential, T_e is the electron temperature in °K and C_1 is a constant involving the number density of neutral atoms and other well-known constants.

Using (17) in (1), one obtains

$$N_e = N_0\left\{1+\left(\frac{3}{4}+\frac{U}{2kT_0}\right)\left(\frac{T_e-T_0}{T_0}\right)\right\}. \tag{18}$$

This equation suggests that the above analysis is applicable to this particular case when $(\partial N_e/\partial T_e)_{T_e=T_0}$ is replaced by $\{(N_0/T_0)(3/4+U/2kT_0)\}$.

To be able to appreciate the effect of the above non-linearity, we consider a case in which only the extraordinary modes of the two waves are propagating and the ordinary modes are absent, i.e.

$$\epsilon_{1x}-\mathrm{i}\epsilon_{1y}=0 \qquad \text{or} \qquad \epsilon_{1x}=\mathrm{i}\epsilon_{1y}$$

and

$$\epsilon_{2x}-\mathrm{i}\epsilon_{2y}=0 \qquad \text{or} \qquad \epsilon_{2x}=\mathrm{i}\epsilon_{2y}.$$

The solution of (14) for such a case can be expressed in the form

$$\begin{aligned}(\epsilon_{1x}+\mathrm{i}\epsilon_{1y}) = (\epsilon_{1x0}+\mathrm{i}\epsilon_{1y0})\exp(\mathrm{i}\beta_1{}^+\xi)[1+\alpha\epsilon_{1x0}\tilde{\epsilon}_{1x0}X_1\{1-\exp(-2k_1\xi)\} \\ +\alpha\epsilon_{2x0}\tilde{\epsilon}_{2x0}X_2\{1-\exp(-2k_2\xi)\}]\end{aligned} \tag{19}$$

where the expressions for X_1 and X_2 are given in the appendix. As the wave propagates large distances ($\xi \to \infty$) in the medium, the ratio Θ of the non-linear to the linear part is given by

$$\Theta = \alpha(\epsilon_{1x0}\tilde{\epsilon}_{1x0}X_1+\epsilon_{2x02}\tilde{\epsilon}_{x0}X_2). \tag{19a}$$

Some numerical calculations have been carried out, on an IBM 1620 computer, to investigate the dependence of the magnitudes of X_1 and X_2 on the various relevant parameters.

Tables 1 and 2 show the variations in the magnitudes of X_1 and X_2 respectively, with the electron density and the electron collision frequency for $s = 1$, $G = 1$, $\omega_0 = \omega_1$, $U/kT_0 = 10$, $\omega_2/\omega_1 = 3$ and for the case of gyro-resonance of the first wave ($\omega_B = \omega_1$). The figures written in the brackets, in all the tables given here, represent the variations of X_1 and X_2 with the various parameters in the absence of the non-linearity discussed in this paper, i.e. with $(\partial N_e/\partial T_e)_{T_e=T_0}$ in equation (1) equal to zero. The figures in table 1 suggest that the magnitude of X_1 decreases with increasing electron density and electron collision frequency; the decrease in the magnitude of X_1 is greater with the increasing electron collision frequency than with increasing electron density. These figures further suggest that the magnitude of X_1 is always appreciably higher when the dependence of N_e on T_e is taken into account than in the absence of this non-linearity. The nature of the variation of X_2 with these parameters is parallel to that of X_1, but the magnitude of X_2, for the same parameters, is greater than the magnitude of X_1.

Table 1. Variation of the magnitude of X_1 with the electron density and electron collision frequency

$(\omega_p/\omega_0)^2$	X_1 for $\nu_0/\omega_0 = 0{\cdot}04$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}06$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}06$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}10$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}12$ 10^{-2}
0·2	8·9440 (0·8518)	4·3225 (0·4122)	2·6235 (0·2499)	1·7904 (0·1705)	1·3130 (0·1250)
0·4	8·1336 (0·7746)	3·7950 (0·3614)	2·2360 (0·2130)	1·4952 (0·1424)	1·0821 (0·1031)
0·6	7·8651 (0·7491)	3·6149 (0·3443)	2·1009 (0·2001)	1·3878 (0·1322)	0·9938 (0·0946)
0·8	7·7318 (0·7364)	3·5253 (0·3358)	2·0334 (0·1937)	1·3337 (0·1270)	0·9488 (0·0904)

$U/kT_0 = 10$, $\omega_2/\omega_1 = 3$ and $\omega_B = \omega_1$. The figures in parentheses denote the variation of X_1 without taking into account the dependence of N_e on T_e.

Table 2. Variation of the magnitude of X_2 with the electron density and electron collision frequency for $\omega_B = \omega_1$

$(\omega_p/\omega_0)^2$	X_2 for $\nu_0/\omega_0 = 0{\cdot}04$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}06$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}08$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}10$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}12$ $\times 10^{-3}$
0·2	24·8150 (5·2215)	8·8135 (1·8533)	4·1671 (0·8755)	2·3172 (0·4863)	1·4243 (0·2985)
0·4	17·3460 (3·6501)	6·2517 (1·3146)	3·0221 (0·6349)	1·7143 (0·3598)	1·0762 (0·2255)
0·6	13·9130 (2·9276)	5·0333 (1·0584)	2·4436 (0·5134)	1·3930 (0·2923)	0·8790 (0·1842)
0·8	11·8280 (2·4889)	4·2833 (0·9007)	2·0818 (0·4374)	1·1888 (0·2495)	0·7515 (0·1575)

Tables 3 and 4 illustrate the dependence of the magnitudes of X_1 and X_2 respectively on the electron density and the electron collision frequency, for the same parameters as in the above tables, for the general case $\omega_1 \neq \omega_B \neq \omega_2$ and for $\omega_B/\omega_0 = 0{\cdot}2$, i.e. for a weak magnetic field. The figures in table 3 show that the magnitude of X_1 decreases with an increase in the electron density and the electron collision frequency; the decrease in the magnitude of X_1 is steeper with the increasing electron collision frequency than with the electron density. The magnitude of X_2, as shown by the figures in table 4, decreases with increasing electron collision frequency, while it increases with an increase in the electron density.

Table 3. Variation of the magnitude of X_1 with the electron density and electron collision frequency

(ω_p^2/ω_0^2)	X_1 for $\nu_0/\omega_0 = 0{\cdot}04$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}06$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}08$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}10$ $\times 10^{-2}$	$\nu_0/\omega_0 = 0{\cdot}12$ $\times 10^{-2}$
0·2	16·0150 (0·1054)	7·1964 (0·0715)	4·1101 (0·0549)	2·6820 (0·0453)	1·9067 (0·0392)
0·4	15·9740 (0·1051)	7·1552 (0·0711)	4·0679 (0·0544)	2·6386 (0·446)	1·8617 (0·0382)
0·6	15·6280 (0·1028)	6·8230 (0·0678)	3·7540 (0·0502)	2·3449 (0·0396)	1·5892 (0·0326)
0·8	6·9552 (0·0458)	3·0806 (0·0306)	1·7349 (0·0232)	1·1167 (0·0189)	0·7836 (0·0161)

$\omega_B/\omega_0 = 0{\cdot}2$ (for $\omega_1 \neq \omega_B \neq \omega_2$).

Table 4. Variation of the magnitude of X_2 with the electron density and electron collision frequency for $\omega_B/\omega_0 = 0{\cdot}2$

(ω_p^2/ω_0^2)	X_2 for $\nu_0/\omega_0 = 0{\cdot}04$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}06$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}08$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}10$ $\times 10^{-3}$	$\nu_0/\omega_0 = 0{\cdot}12$ $\times 10^{-3}$
0·2	6·6928 (0·0482)	2·9305 (0·0319)	1·6395 (0·0240)	1·0675 (0·0194)	0·7535 (0·0169)
0·4	7·8448 (0·0565)	3·4865 (0·0379)	1·9738 (0·0289)	1·2824 (0·0237)	0·9063 (0·0203)
0·6	10·7490 (0·0775)	4·7431 (0·0516)	2·6584 (0·0389)	1·6951 (0·0313)	1·1718 (0·0262)
0·8	19·2310 (0·1386)	7·0054 (0·0762)	3·4349 (0·0502)	1·9853 (0·0366)	1·2735 (0·0285)

To be able to appreciate the effect of this non-linearity (i.e. the dependence of N_e on T_e) on the magnitude of the ratio Θ of the non-linear to the linear part in (19*a*), let us consider the following cases for

$$T_0 = 3000\,°\mathrm{K}, \qquad \omega_0 = 10^6\,\mathrm{Hz},$$
$$E_{1x0} = E_{2x0} = 4{\cdot}4\times 10^{-5}\,\mathrm{v\,cm^{-1}},$$
$$(\omega_p/\omega_0)^2 = 0{\cdot}6 \quad \text{and} \quad \nu_0/\omega_0 = 0{\cdot}10.$$

(i) Considering the case when $\omega_B = \omega_1$ using (19*a*) and tables 1 and 2 we get

$$\Theta = 0{\cdot}1529$$

in the presence of this non-linearity while

$$\Theta = 0{\cdot}0301$$

when the dependence of N_e on T_e is not taken into account.

(ii) For the case when $\omega_1 \neq \omega_B \neq \omega_2$, one obtains, using (19*a*) and tables 3 and 4,

$$\Theta = 0{\cdot}1917$$

(taking the dependence of N_e on T_e into account) and

$$\Theta = 0{\cdot}0035$$

(when the dependence of N_e on T_e is neglected).

This suggests that the contribution, due to this non-linearity, to the non-linear part of the amplitude of the wave, may be as much as about 20%.

An expression for X_1 and X_2, without the presence of a magnetic field in a plasma, can be obtained from those discussed above, for the case $\omega_1 \neq \omega_B \neq \omega_2$, by putting $\omega_B = 0$ and $E_{1y} = E_{2y} = 0$. Some numerical calculations for the magnitudes of X_1 and X_2 have been carried out for this case as well, but they have not been included here owing to lack of space. The nature of the variation of X_1 and X_2 with the various parameters is similar to that brought out by the last two tables, but the relative magnitudes of X_1 and X_2 are small compared to those for the case $\omega_1 \neq \omega_B \neq \omega_2$.

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Appendix

The various coefficients of equation (19) are given by the following expressions, for the case when the frequency of the first wave is equal to the gyrofrequency of electrons:

$$X_1 = \frac{(\omega_p/\omega_0)^2 Y^3\{(3/4+U/2kT_0)-1/2\}}{3\sqrt{\pi}k_1(n_1{}^2+4k_1{}^2)^{1/2}}\exp(i\psi_1) \qquad \text{where} \qquad \tan\psi_1 = \frac{2k_1}{n_1}$$

$$X_2 = \frac{(\gamma^2+\delta^2)^{1/2}\exp(i\psi_2)}{4k_2\{n_1{}^2+(k_1+k_2)^2\}^{1/2}}, \qquad \tan\psi_2 = \frac{n_1\delta-(k_1+k_2)\gamma}{n_1\gamma+(k_1+k_2)\delta}, \qquad Z = \left(\frac{\nu_0}{\omega_0}\right) = \frac{1}{Y}$$

$$\gamma = I_2+\tfrac{1}{2}H_2(Y+A_2)+\tfrac{1}{2}F_2B_2, \qquad \delta = J_2-\tfrac{1}{2}H_2B_2+\tfrac{1}{2}F_2(Y+A_2)$$

$$A_2 = \frac{1}{\{(\omega_1/\omega_2)^2-1\}^2}\left[6Z^3\left(\frac{\omega_1}{\omega_2}\right)^4+2Z\left(\frac{\omega_1}{\omega_2}\right)^2\left\{1+\left(\frac{\omega_1}{\omega_2}\right)^2\right\}+4Z\left(\frac{\omega_1}{\omega_2}\right)^3\right]$$

$$B_2 = \frac{3\sqrt{\pi}(\omega_1/\omega_2)}{4\{(\omega_1/\omega_2)^2-1\}^2}\left[\left(1+\frac{\omega_1}{\omega_2}\right)\left\{\left(\frac{\omega_1}{\omega_2}\right)^2-1\right\}-\frac{5}{2}\left(1-\frac{\omega_1}{\omega_2}\right)Z^2\left(\frac{\omega_1}{\omega_2}\right)^2\right]$$

$$H_2 = \frac{4(\omega_p/\omega_0)^2}{3\sqrt{\pi}\{(\omega_1/\omega_2)^2-1\}^2}\left[9Z^2\left(\frac{\omega_1}{\omega_2}\right)^4+\left(\frac{\omega_1}{\omega_2}\right)^2\left\{1+\left(\frac{\omega_1}{\omega_2}\right)^2\right\}+\left(\frac{\omega_1}{\omega_2}\right)^3\right]$$

$$F_2 = \frac{5(\omega_p/\omega_0)^2Z(\omega_1/\omega_2)^3(\omega_1/\omega_2-1)}{\{(\omega_1/\omega_2)^2-1\}^2}$$

$$I_2 = \frac{8(\omega_p/\omega_0)^2\{(3/4+U/2kT_0)-1\}}{3\sqrt{\pi}\{(\omega_1/\omega_2)^2-1\}^2}\left[3Z\left(\frac{\omega_1}{\omega_2}\right)^4+\left(\frac{\omega_1}{\omega_2}\right)^2\left\{1+\left(\frac{\omega_1}{\omega_2}\right)^2\right\}+2\left(\frac{\omega_1}{\omega_2}\right)^3\right]$$

and

$$J_2 = \frac{(\omega_p/\omega_0)^2Y^2(\omega_1/\omega_2)\{(3/4+U/2kT_0)-1\}}{\{(\omega_1/\omega_2)^2-1\}^2}\left[\left(1+\frac{\omega_1}{\omega_2}\right)\left\{\left(\frac{\omega_1}{\omega_2}\right)^2-1\right\}\right.$$
$$\left.-\frac{5}{2}\left(1-\frac{\omega_1}{\omega_2}\right)Z^2\left(\frac{\omega_1}{\omega_2}\right)^2\right].$$

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